Ambiguity and Financial Uncertainty in a Real Business Cycle Model

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Abstract

Financial uncertainty measured by the risk neutral variance is negatively related to consumption, investment, output and the price-dividend ratio but positively related to future stock returns and volatilities of consumption growth, investment growth, output growth and stock returns. In addition, the mean and volatility of the variance risk premium are large and cannot be explained by standard asset pricing models. We examine a production-based asset pricing model where productivity growth follows a Markov process with time varying conditional mean and volatility and the representative agent has ambiguity aversion preferences. When the model is calibrated to match unconditional moments of macroeconomic quantities and asset returns, and the dividends dynamics are calibrated to be procyclical, the model can reproduce the relations between the risk neutral variance and both the level and variation of quantities and returns observed in the data. The model can also generate a sizable variance risk premium.

JEL Classification: C61; D81; G11; G12.

Keywords: Ambiguity aversion, business cycle, production economy, risk neutral variance, variance risk premium.

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1 Introduction

Financial uncertainty has drawn growing attention in recent macro-finance research. The most common measure of financial uncertainty is the Chicago Board Option Exchange’s volatility index (VIX). Because VIX is a measure of the implied volatility of S&P 500 index options, it gauges investors’ fear of economic uncertainty. In the finance literature, the risk neutral volatility reflects the representative agent’s concern about macroeconomic uncertainty (Bollerslev et al. (2009), Drechsler and Yaron (2011) and Drechsler (2013)), or alternatively the agent’s time varying risk aversion (Rosenberg and Engle (2002) and Bollerslev et al. (2008)). Empirical evidence has shown that financial uncertainty proxied by the risk neutral volatility leads to declines in economic activity and the stock market valuation, see Bloom (2009) and Gourio (2012). Moreover, investors seek compensation by trading equity index options to hedge against the risk of high stock market volatility, giving rise to variance risk premium. Variance risk premium is defined as the difference between the expected stock return variances under the risk-neutral and physical measures. Several important stylized facts about the market variance risk premium include: (1) both the average and standard deviation of variance risk premium are large and cannot be explained by standard asset pricing models, leading to “variance risk premium puzzle”, (2) the empirical distribution of variance risk premium has non-zero skewness and high kurtosis, and (3) variance risk premium can forecast market returns at short-to-medium horizons, see Bollerslev et al. (2009), Drechsler and Yaron (2011), and Drechsler (2013) for more discussion.

In this paper, we study a real business cycle (RBC) model with time varying uncertainty and ambiguity aversion to gain clear understanding in the dynamics of the risk neutral variance and its relations with macroeconomic fundamentals. Furthermore, we explore the extent of this model in explaining variance risk premium in the data. We consider a RBC model with stochastic productivity growth. The productivity growth rate is assumed to follow a Markov process with time varying conditional mean and volatility. The representative agent is assumed to be ambiguity averse in the spirit of Klibanoff et al. (2005) and Ju and Miao (2012). In the agent’s view, the physical transition probabilities governing the Markov chain are distorted in an endogenous and pessimistic way such that states with low continuation value receive more weight than in the traditional, ambiguity-neutrality case with Epstein-Zin recursive preferences (Epstein and Zin (1989)). With a relative risk aversion parameter of 1.5 and an elasticity of intertemporal substitution greater than
1, the model can match reasonably well salient features of macroeconomic quantities and asset returns in the data, namely, a low volatility of consumption growth, a high volatility of investment growth, correlations among growth rates of consumption, investment and output, low and smooth risk-free rates and high equity premium. More importantly, the model generates negative relations between the risk neutral variance, which measures financial uncertainty, and aggregate quantities. In addition, the model implies that the risk neutral variance is negatively associated with the equity value and carries positive risk premium. Moreover, in the model, the risk neutral variance has significantly positive correlations with volatilities of quantities and the stock market volatility at different leads and lags. We find that all these model implications are consistent with empirical observations in the data. The model can also generate a sizable variance risk premium that is about half of the data. This is a remarkable result because we show that the standard IID growth model without ambiguity aversion implies a variance risk premium close to 0, even with very high risk aversion. The distribution of the variance risk premium implied by the model is non-Gaussian with high kurtosis.

Previous research typically uses the endowment economy framework to analyze the risk neutral volatility and variance risk premium, for example, Drechsler (2013) and Miao et al. (2012). The dynamics of consumption and dividends are exogenously specified in these models. They rely on time varying uncertainty in the form of Knightian uncertainty (ambiguity) or parameter and state uncertainty to replicate the dynamics of the risk neutral variance in the data. On the other hand, the dynamic stochastic general equilibrium (DSGE) model provides a natural and unified framework to study the joint behavior of the business cycle variables and asset prices. However, DSGE models generally face a great challenge in explaining important features of macroeconomic quantities and asset prices once consumption and dividends are endogenously determined (Jermann (1998) and Kaltenbrunner and Lochstoer (2010)). Until now, a coherent analysis of the risk neutral volatility and the business cycle is still lacking in the literature. To our knowledge, the contribution of this paper is to first provide such an analysis.

To build macroeconomic uncertainty in the model, we follow Liu and Miao (2015) and assume that the conditional mean and volatility of aggregate productivity growth follow two independent two-state Markov chains. We use the post-war total factor productivity (TFP) data to estimate the Markov-switching model and find that the high expected growth regime is very persistent while the low expected growth regime is transitory, and that both volatility regimes are quite persistent.
The postulated dynamics of productivity growth imply a set of possible conditional distributions of productivity growth. The model distinguishes risk from subjective uncertainty, where risk is characterized by a conditional distribution of the growth rate, and subjective uncertainty, which is alternatively interpreted by Klibanoff et al. (2005, 2009) as ambiguity, is summarized by the agent’s belief over the productivity growth regimes.

Following Ju and Miao (2012), we assume that the representative agent has generalized recursive smooth ambiguity preferences axiomatized by Hayashi and Miao (2011). This class of preferences allow for a three-way separation among risk aversion, ambiguity aversion and the elasticity of intertemporal substitution (EIS). Recursive preferences in the spirit of Epstein and Zin (1989) are a special case when the agent is ambiguity neutral. The ambiguity-averse agent has a pessimistic view on productivity growth states and puts more weight on states with low continuation value. We show that ambiguity aversion effectively distorts the physical transition probabilities of the Markov chain for the conditional mean and volatility of productivity growth. The distortion is endogenous and its size depends on the state of the economy. The endogenous pessimism greatly magnifies the impact of macroeconomic uncertainty on asset prices and generates important asset pricing implications. In particular, high market price of risk implies a high equity risk premium, and time-varying pessimism induces a high risk neutral variance, resulting in a high variance risk premium. We use the method of detection-error probabilities following Anderson et al. (2003) and Jahan-Parvar and Liu (2014) to quantify the extent of ambiguity aversion.¹

To better match data features of asset returns and particularly equity volatility, we calibrate the representative firm’s payouts (cashflows) to the aggregate dividends data. Following the consumption-based asset pricing literature (Abel (1999), Bansal and Yaron (2004) and Ju and Miao (2012), among others), aggregate dividends are assumed to be a levered claim on aggregate consumption. Unlike the consumption-based framework, since consumption is endogenous in the RBC model, dividends are also endogenous. The advantage of imposing procyclical cashflows is that the equity claim becomes sufficiently risky. RBC models typically imply countercyclical payouts. In times of high productivity growth, investment is high and thus the firm’s payouts tend to be low. Because the firm’s payouts are countercyclical, much of risk in the equity claim is removed.

¹In the ambiguity literature, the smooth ambiguity approach differs from the multiple priors approach (for example, Gilboa and Schmeidler (1989) and Chen and Epstein (2002)) in that the multiple priors model features the max-min approach, which implies a tight link between ambiguity and ambiguity aversion. The smooth ambiguity model unravels this tight link and thus allows for comparative statics analysis by holding the set of candidate models fixed, while varying the extent of ambiguity aversion.
leading to counterfactually low volatility of equity returns.\footnote{Favilukis and Lin (2013) show that a model with wage rigidity in the form of infrequent negotiation can generate procyclical variation in the firm’s payouts.}

The quantitative analysis shows that under ambiguity aversion, regime switching in the conditional mean of productivity growth has a large impact on the stochastic discount factor, equity risk premium and risk neutral variance. However, the impacts of volatility regime switching on quantities and moments of asset returns are relatively small. A decrease in expected productivity growth implies that the optimal level of capital becomes permanently lower, and that both investment and consumption fall toward a lower trend. The decline in investment leads to lower expected consumption growth. As the state of productivity worsens, marginal utility of an ambiguity-averse agent rises substantially more than for an ambiguity-neutral agent with Epstein-Zin recursive utility. The induced pessimism increases the stochastic discount factor significantly, leading to a lower price-dividend ratio, higher equity premium and volatility. Moreover, the pessimism driven by ambiguity aversion largely raises the risk-neutral expected variance and therefore the variance risk premium. Thus, the model implies that financial uncertainty proxied by the risk neutral variance is negatively priced and carries positive risk premium, in line with the consumption-based results found by Bollerslev et al. (2009). Naturally, the model also implies that the risk neutral variance is negatively related to the level of aggregate quantities.

When the volatility of productivity growth shifts to the high state, both the risk neutral variance and variance risk premium rise, albeit moderately. This is because the volatility regime switching leads to a decline in investment but an increase in current consumption. While the former effect tends to raise the stochastic discount factor through the long-run risk channel, the latter has an opposite impact on the stochastic discount factor. In addition, ambiguity aversion has limited scope in altering the stochastic discount factor in the case of volatility regime switching as opposed to mean switching. This feature is in contrast to Liu and Miao (2015), where disappointment aversion makes the volatility risk significantly priced. In our model, the presence of persistent volatility regime switching in productivity is sufficient to reproduce the positive relation between the risk neutral variance and volatilities of quantities growth rates and equity returns.

This paper is connected with the production-based asset pricing literature. Production-based asset pricing attracts growing interests in recent research. Kaltenbrunner and Lochstoer (2010) study implications of recursive preferences on quantities and asset returns in DSGE models with
autoregressive productivity levels and IID productivity growth respectively. Croce (2010) considers long run productivity risk and finds that with an EIS greater than 1, the model can match salient features of quantities and asset returns well. Gourio (2012) examines the impact of time-varying disaster risk in a RBC model. Liu and Miao (2015) assume the same Markov switching process for productivity growth as in this paper and consider generalized disappointment aversion preferences. They find that disappointment aversion makes the productivity volatility risk significantly priced with procyclical dividends. In addition, their model can endogenously reproduce long run volatility risk in consumption growth. Favilukis and Lin (2013) examine a DSGE model with labor market friction. The model can explain several important asset pricing puzzles including a high and volatile equity premium, value premium, and the term structure of equity returns. Similar to this paper, Jahan-Parvar and Liu (2014) investigate ambiguity aversion and learning in a production economy with regime switching productivity growth. They find that learning under ambiguity aversion is important for asset pricing, i.e., generating a high equity premium, volatility clustering and kurtosis of equity returns, and the predictability of returns by key macroeconomic and financial variables. Backus et al. (2015) also analyze implications of ambiguity aversion in RBC models.

The analysis of the risk neutral variance and variance risk premium in equilibrium settings has so far been confined to endowment economies. Bollerslev et al. (2009) and Zhou (2010) emphasize the importance of stochastic economic uncertainty, i.e., stochastic vol-of-vol in the consumption growth process. Drechsler (2013) consider jump shocks to cash flow growth and volatility and assume that the representative agent worries about model misspecification. Based on the multiple priors approach, the model can explain the magnitude and fluctuation of the variance risk premium observed in the data. Miao et al. (2012) use the same preferences as in this paper to analyze ambiguity about consumption growth states and the variance risk premium. Bianchi (2014) examines real-time learning on state variables and parameters in the consumption growth process.

The rest of the paper is organized as follows. Section 2 provides empirical evidence of the impact of the variance risk premium on quantities and asset prices. Section 3 presents a benchmark model with smooth ambiguity preferences and Markov switching productivity growth rates. Section 4 calibrates the model to the historical data, discusses quantitative results and performs sensitivity analysis. Section 5 concludes. The numerical algorithm and the data construction are described in the Appendix.
2 Empirical Analysis

2.1 Data construction

The market variance risk premium, as a proxy for economic uncertainty, is not directly observable but can be estimated from the difference between model-free implied variance and the conditional expectation of realized variance. Formally, the variance risk premium is defined as the difference between the risk-neutral expectation and the objective expectation of the return variance,

\[ VRP_t \equiv E^Q_t [\sigma_{R,t+1}^2] - E_t [\sigma_{R,t+1}^2] \]

where \( E^Q_t [\cdot] \) denotes the risk-neutral expectation, \( E_t [\cdot] \) is the physical expectation, and \( \sigma_{R,t+1}^2 \) is the return variance.

We compute the risk-neutral expected variance \( E^Q_t [\sigma_{R,t+1}^2] \) following the volatility index (VIX) method used by the Chicago Board of Options Exchange (CBOE),

\[ E^Q_t [\sigma_{R,t+1}^2] = \frac{2}{T} \sum_i \frac{\Delta K_i}{K_i^2} e^{r_f T} Q(K_i, T) - \frac{1}{T} \left[ \frac{F}{K_0} - 1 \right]^2, \]

where \( T \) is time to maturity, \( K_i \) is the strike price of the \( i \)th out-of-the-money option, \( r_f \) is the LIBOR rate for maturity \( T \), \( Q(K_i, T) \) is the midpoint of bid and ask prices for option with maturity \( T \) and strike \( K_i \), \( F \) is the forward index level derived from index option prices, and \( K_0 \) is the first strike below \( F \). The objective expectation \( E_t [\sigma_{R,t+1}^2] \) is estimated from daily S&P500 index (SPX) returns. The LIBOR yield curve, S&P500 index returns and options data are from OptionMetrics and the sample spans from January 1996 to December 2012. Because real business cycle models are often calibrated at a quarterly frequency, we construct quarterly market VRP data. In particular, every quarter end, we compute the risk neutral variance with SPX index option prices for the maturity on either side of 90 days and linearly interpolate the risk neutral variance to obtain the 90-day risk neutral variance. To estimate the objective expectation of the quarterly variance, we simply compute the sum of squared daily returns of the quarter and assume that the realized variance is used as the objective expectation \( E_t [\sigma_{R,t+1}^2] \).

Figure 1 plots the constructed quarterly risk neutral variance and variance risk premium. We find that the risk-neutral variance peaks during the 1998 Asian-Russian crises, 2001 “dot-com” bub-

\footnote{For robustness, we also compute the sum of squared log daily returns of the quarter and the results are almost identical.}
able, and recent 2008 financial crisis followed by heightened economic uncertainty due to sovereign
debt issues. The variance risk premium is obtained by subtracting the impact of the objective
expectation of variance. During the recent financial crises, the variance risk premium exhibited
a significant drop because of a remarkable increase in realized variance. The full sample average
of VRP is 12.08 (in percentage squared, monthly terms), close to the moment reported in other
papers (Drechsler (2013) and Miao et al. (2012)). The average VRP (15.15) is slightly higher for
the sample period prior to the financial crisis. The full sample standard deviation is 34.95, which
is higher than the pre-crisis moment (17.34). This indicates that a large fraction of variation in
VRP occurs during the crisis period. In the full sample, the negative skewness (-4.97) and high
excess kurtosis (35.31) show that the distribution of VRP is not Gaussian. This is also true for
the pre-crisis period, though the skewness for the pre-crisis sample is positive (2.76) and the excess
kurtosis is much smaller (10.82) than in the full sample.

Insert Figure 1 about here

2.2 Expected productivity growth and the risk neutral variance

Several papers have analyzed the impact of uncertainty shocks on the business cycle. This liter-
ature considers different measures of uncertainty including the VIX measure (Bloom (2009)), the
dispersion of total factor productivity shocks to plants and establishments (Bloom et al. (2013)),
survey data (Leduc and Liu (2013)), and time varying volatility of TFP growth (Liu and Miao
(2015)). All of these studies find that an increase in uncertainty dampens economic activity and
leads to a decline in consumption, investment and output.

Naturally, our constructed risk neutral variance is a proxy for financial uncertainty. Our goal
here is to investigate the empirical relation between the risk neutral variance and the business
cycle fluctuation. More specifically, we want to establish the link between expected productivity
growth and the risk neutral variance and therefore attribute the financial uncertainty proxy and
its negative impacts on quantities to changes in expected productivity growth.

We denote by $\Delta a_t$ the productivity growth rate, i.e., $\Delta a_t \equiv \ln(A_t/A_{t-1})$ where $A_t$ is the
productivity level. We assume that productivity growth follows a Markov switching process as in
Liu and Miao (2015). The Markov-switching model has independent mean and volatility regime
\[ \Delta a_t = \mu(s_t^\mu) + \sigma(s_t^\sigma) \epsilon_t, \quad \epsilon_t \sim N(0,1), \]  

where \( s_t^\mu \) and \( s_t^\sigma \) determine the regimes of the conditional mean and volatility of the growth rate respectively. The same specification has been used by Lettau et al. (2008) and Kuehn et al. (2013) to model consumption growth dynamics. The expected productivity growth rates in the expansion and recession regimes are denoted by \( \mu_h \) and \( \mu_l \) respectively. The two states for the conditional volatility are \( \sigma_l \) and \( \sigma_h \) with \( \sigma_h > \sigma_l \). Since regime shifts for the conditional mean and volatility are independent, the transition matrix can be characterized by four transition probabilities:

\[
P(s_t^\mu = 0 | s_{t-1}^\mu = 0) = p_{l\mu}, \quad P(s_t^\mu = 1 | s_{t-1}^\mu = 1) = p_{h\mu},
\]

\[
P(s_t^\sigma = 0 | s_{t-1}^\sigma = 0) = p_{l\sigma}, \quad P(s_t^\sigma = 1 | s_{t-1}^\sigma = 1) = p_{h\sigma}.
\]

We use macroeconomic data to construct Solow residuals and quarterly productivity growth rates. The sample period for empirical estimation of the Markov-switching process is 1952:Q1—2012:Q4. Details of data construction are in the Appendix. The Markov switching model is estimated using the expectation maximization algorithm developed by Hamilton (1990). The parameter estimates are summarized in Table 1. The high (low) mean growth estimate is 0.35% (-1.2%) per quarter. The transition probabilities estimates indicate that the high mean growth regime is persistent \((p_{h\mu} = 0.97)\) but the low mean growth regime is transitory \((p_{l\mu} = 0.59)\). In addition, the high volatility regime \(\sigma_h = 1.82\%\) features significantly more variation than the low volatility regime \(\sigma_l = 0.75\%\). Both volatility regimes are persistent with transition probabilities estimates close to 1. The estimation results are similar to those reported in other papers such as Cagetti et al. (2002) and Liu and Miao (2015).

[Insert Table 1 about here]

Figure 2 plots time series of conditionally expected productivity growth, which are obtained from the estimates and filtered probabilities of the growth regimes. Compared to productivity growth, its conditional mean is a slow moving process with declines during economic recessions. This persistent component captures long run productivity risk, which has been explored in Croce (2010). We ask the question: does a decline in expected productivity growth lead to more financial uncertainty proxied by the risk neutral variance, which subsequently causes declines in macroeconomic quantities and equity valuation? To answer this question, we perform a Vector
Autoregressive (VAR) analysis with seven endogenous variables including conditionally expected productivity growth, the risk neutral variance, gross domestic product (GDP), consumption, investment, hours worked, and the price-dividend ratio. All variables except for conditionally expected productivity growth are in logarithms. Macroeconomic data are drawn from the National Income and Product Accounts (NIPA). Consumption, investment and GDP data are deflated by the corresponding deflators. Financial data are drawn from the Center for Research in Security Prices (CRSP). The price-dividend ratio data are constructed from value-weighted index returns including and excluding distributions. We use the Bayesian approach developed by Sims and Zha (1998) to estimate the VAR model. Because the sample starts from 1996:Q1, we consider a lag of 1 in the estimation.

Figure 3 presents the impulse responses of other variables to a negative one-standard-deviation shock to the conditional mean of productivity growth, obtained from the Bayesian VAR estimation. The figure shows that investment, consumption, output and hours worked drop on impact, while the risk neutral variance rises following the bad news on expected productivity growth. These empirical results are consistent with Bloom (2009) and Gourio (2012) and suggest that in times of high financial uncertainty, the aggregate consumption and output are low. This implies high marginal utility in the perspective of the representative agent. Moreover, because investment decreases when the risk neutral variance rises, expected future consumption is low, leading to smaller future cash flows and thus lower equity valuation. Figure 3 also indicates that the price-dividend ratio falls in response to the negative shock to expected productivity growth and that a high risk neutral variance is associated with a low price-dividend ratio.

[Insert Figure 3 about here]

Figure 4 plots the correlation between the risk neutral variance and output, consumption, investment and the equity return respectively at different leads and lags, i.e., \( \text{Corr}(E^Q_t \left[ \sigma^2_{R,t+1} \right], x_{t+k}) \) for \( k = -4 \) to 4. Output, consumption, and investment are in logarithms and detrended using HP filter. For aggregate output and consumption, these correlations are negative for \( k < 0 \), suggesting that consumption and output negatively lead the risk neutral variance. In other words, good times with high consumption and output tend to lessen financial uncertainty in the future. On the other hand, we also use the ordinary least square (OLS) method to estimate the VAR model, where the GDP, investment, consumption, and hours worked are detrended using HP (Hodrick and Prescott (1997)) filter. The impulse responses results are similar.
hand, investment positively leads the risk neutral variance up to $k = -1$. More interestingly, the correlations between the risk neutral variance and quantities including consumption, investment and output become negative for $k > 0$. This result implies that financial uncertainty tends to dampen future real economic activity. The lead-lag relations between the risk neutral variance and the equity return suggest that the stock market performance negatively leads the level of financial uncertainty, i.e., high past returns lead to low current risk neutral variance. Moreover, Panel D of Figure 4 shows that high current risk neutral variance forecasts high future returns. This result is consistent with the negative response of the price-dividend ratio to a bad shock to expected productivity growth, which also raises the risk neutral variance. Low current price-dividend ratio implies high future returns. Thus, our empirical result explains the finding of Bali and Zhou (2014) that uncertainty proxied by VIX is negatively related to equity valuation and carries a positive premium.

To examine the link between financial uncertainty and macroeconomic uncertainty, we estimate an AR(1)-GARCH(1,1) model for output growth, consumption growth and investment growth. We then extract the estimated conditional volatility series for each variable and use conditional volatility as a proxy for macroeconomic uncertainty. Figure 5 plots the correlation between the risk neutral variance and volatilities of output growth, consumption growth and investment growth at different leads and lags, i.e., $\text{Corr}(VRP_t, x_{t+k})$ for $k = -4$ to 4. It is noteworthy that for $k > 0$, this correlation is significantly positive, suggesting that the risk neutral variance positively forecasts future volatility of quantities. The correlation between the risk neutral variance and the stock market return volatility has a similar pattern, as shown in Panel D of Figure 5. In fact, the correlations are also positive for $k \leq 0$, indicating that high macroeconomic uncertainty also forecasts high financial uncertainty. Our empirical results suggest that the relation between the two types of uncertainty is very strong. In the next section, we present a production-based general equilibrium model to explain all the important stylized facts documented above.
3 The Model

3.1 Preferences

We assume that productivity growth follows the dynamics specified in Section 2. Suppose that $z_t$ represents the state of productivity growth and determines a conditional distribution of the growth rate at time $t$. The productivity growth dynamics imply that $z_t$ follows a Markov chain. Multiple conditional distributions at each point of time capture the notion of ambiguity, see Klibanoff et al. (2005). Following Ju and Miao (2012), we assume that the representative agent has generalized recursive smooth ambiguity preferences

$$
V_t = \left((1-\beta)U_t^{1-\frac{1}{\psi}} + \beta \left( \mathbb{E}_{z_t} \left[ \left( \mathbb{E}_{z_{t+1}, t} \left[ V_{t+1}^{1-\gamma} \right] \right)^{\frac{1-\eta}{1-\psi}} \right] \right)^{\frac{1}{1-\psi}} \right)^{1-\frac{1}{\psi}} \tag{2}
$$

where the felicity function $U_t$ depends on consumption ($C$) and labor hours ($N$) in the standard Cobb-Douglas form

$$
U_t = C_t (1 - N_t)^\nu.
$$

Regarding preference parameters, $\beta \in (0, 1)$ is the subjective discount factor, $\gamma$ is the coefficient of relative risk aversion, $\eta$ is the ambiguity aversion parameter, and $\psi$ is the elasticity of intertemporal substitution (EIS). Different from recursive preferences with ambiguity neutrality, the certainty equivalent of smooth ambiguity utility is defined by

$$
\mathcal{R}_t (V_{t+1}) = \left( \mathbb{E}_{z_t} \left[ \left( \mathbb{E}_{z_{t+1}, t} \left[ V_{t+1}^{1-\gamma} \right] \right)^{\frac{1-\eta}{1-\psi}} \right] \right)^{\frac{1}{1-\eta}} \tag{3}
$$

where $\mathbb{E}_{z_{t+1}, t} [\cdot]$ denotes the expectation conditional on the history up to time $t$ and a probability distribution of productivity growth in state $z_{t+1}$. Suppose that productivity growth is conditionally normal, the expectation $\mathbb{E}_{z_{t+1}, t} \left[ V_{t+1}^{1-\gamma} \right]$ can be explicitly written as

$$
\mathbb{E}_{z_{t+1}, t} \left[ V_{t+1}^{1-\gamma} \right] = \int V_{t+1}^{1-\gamma} \frac{1}{\sigma_{t+1} \sqrt{2\pi}} \exp \left( -\frac{(\Delta a_{t+1} - \mu_{t+1})^2}{2\sigma_{t+1}^2} \right) d(\Delta a_{t+1})
$$

where $\mu_{t+1}$ and $\sigma_{t+1}^2$ are, respectively, the conditional mean and variance of the growth rate at time $t + 1$.

For comparison, the certainty equivalent under ambiguity neutrality is based on the predictive
distribution of $\Delta a_{t+1}$ in the following
\[
R_t(V_{t+1}) = \left( E_t \left[ V_{t+1}^{1-\gamma} \right] \right)^{1-\gamma} = \left( \int V_{t+1}^{1-\gamma} p(\Delta a_{t+1}|z_t) d(\Delta a_{t+1}) \right)^{1-\gamma}
\]

The predictive density \( p(\Delta a_{t+1}|z_t) \) is given by
\[
p(\Delta a_{t+1}|\Delta a^t) = \int_Z p(\Delta a_{t+1}|z_{t+1}) p(z_t, z_{t+1}) dz_{t+1}
\]
where \( p(\Delta a_{t+1}|z_{t+1}) \) is the conditional density, and \( p(z_t, z_{t+1}) \) denotes the transition density from state \( z_t \) to \( z_{t+1} \).

The key property of the smooth ambiguity model is that it achieves the separation between ambiguity and ambiguity aversion (Klibanoff et al. (2005)). Ambiguity is gauged by a set of conditional distributions of productivity growth, while ambiguity aversion reflects the agent’s attitudes toward uncertainty about future states of the growth rate. For the utility function (2), the agent is ambiguity averse if and only if \( \eta > \gamma \). In the certainty equivalent (3), the extra curvature in the preferences induced by the condition \( \eta > \gamma \) precludes the compound reduction between the transition density and the conditional distribution of the growth rate.

### 3.2 Production setting

There exist a set of conditional probability distributions for productivity growth, namely, (1) low mean—low volatility regime \( (\mu_l, \sigma_l) \), (2) low mean—high volatility regime \( (\mu_l, \sigma_h) \), (3) high mean—low volatility regime \( (\mu_h, \sigma_l) \), and (4) high mean—high volatility regime \( (\mu_h, \sigma_h) \). Using the notation in the previous section, we have \( z_t = \{s_l^\mu_t, s_l^\sigma_t\} \in Z \). The size of the set of conditional distributions characterizes ambiguity.

Aggregate output \( (Y_t) \) is produced according to standard constant-returns-to-scale Cobb-Douglas production function:
\[
Y_t = K_t^\alpha (A_t N_t)^{1-\alpha}, \quad \alpha \in (0, 1),
\]
where \( \alpha \) is the capital share, and \( K_t \) denotes the capital stock. Capital adjustment is costly. The law of motion for capital accumulation is
\[
K_{t+1} = (1 - \delta) K_t + \phi \left( \frac{I_t}{K_t} \right) K_t,
\]

(4)
and the adjustment cost function is given by (see Jermann (1998))

\[ \phi \left( \frac{I_t}{K_t} \right) = a_1 + \frac{a_2}{1 - 1/\xi} \left( \frac{I_t}{K_t} \right)^{1-1/\xi}, \quad a_2 > 0, \quad \xi > 0. \]

where \( \xi \) is the elasticity of the investment rate to Tobin’s Q, and the parameters \( a_1 \) and \( a_2 \) are chosen such that there is no adjustment cost in the steady state.\(^5\)

### 3.3 Equilibrium characterization

The equilibrium can be characterized by the social planner’s problem.\(^6\) The social planner’s problem is to choose \( \{C_t, I_t, N_t\} \) to maximize (2) subject to the resource constraint \( Y_t = C_t + I_t \) and the law of motion for capital accumulation. In the model, the state variables are \( \{K_t, A_t, s_t^\mu, s_t^\sigma\} \). Denote \( J_t = J(K_t, A_t, s_t^\mu, s_t^\sigma) \) the value function. The Bellman equation can be written as

\[
J_t = \max_{C_t, I_t, N_t} \left[ (1 - \beta) U_t^{1-\psi} + \beta \left( \mathbb{E}_{z_{t+1}} \left[ \left( \mathbb{E}_{z_{t+1}} \left[ J_{t+1}^{1-\gamma} \right] \right)^{1-\eta} \right] \right)^{1-1/\psi} \right]
\]

Due to homogeneity and the fact that \( K_t, C_t, I_t \) and \( Y_t \) have the common trend \( A_t \), we can write

\[
\left\{ \tilde{K}_t, \tilde{C}_t, \tilde{I}_t, \tilde{Y}_t \right\} = \left\{ \frac{K_t}{A_t}, \frac{C_t}{A_t}, \frac{I_t}{A_t}, \frac{Y_t}{A_t} \right\}.
\]

It follows that the value function can be rewritten as

\[
J(K_t, A_t, s_t^\mu, s_t^\sigma) = A_t J(\tilde{K}_t, s_t^\mu, s_t^\sigma)
\]

\(^5\)That is, \( \phi(I/K) = I/K \) and \( \phi'(I/K) = 1 \) in the deterministic steady state. The two equations imply that

\[
a_1 = \frac{\exp(\bar{\mu}) - 1 + \delta}{1 - \xi}, \quad a_2 = \left( \exp(\bar{\mu}) - 1 + \delta \right)^{1/\xi},
\]

where \( \bar{\mu} \) denotes the unconditional mean of the productivity growth rate and is given by

\[
\bar{\mu} = \frac{1 - p_{ll}^\mu}{2 - p_{ll}^\mu - p_{lh}^\mu} \mu_h + \frac{1 - p_{hh}^\mu}{2 - p_{ll}^\mu - p_{lh}^\mu} \bar{\mu}_l.
\]

\(^6\)This economy can be decentralized using the standard arguments: the representative household supplies labor inputs and trades shares issued by the firm and risk-free bonds, and the firm chooses labor and investment to maximize its firm value (the discounted present value of future cash flows).
where $\tilde{J}(\tilde{K}_t, s_t^\mu, s_t^\sigma)$ satisfies the Bellman equation

$$
\tilde{J}_t = \max_{\tilde{C}_t, I_t, N_t} \left[ (1 - \beta) \left( \tilde{C}_t (1 - N_t) \right)^{1 - \frac{1}{\nu}} + \beta \left( \mathbb{E}_{z_t} \left[ \left( \text{exp} \left( \Delta a_{t+1} \right) \tilde{J}_{t+1} \right)^{1 - \gamma} \right] \right)^{\frac{1 - \gamma}{1 - \eta}} \right]^{1 - \frac{1}{\psi}}
$$

subject to

$$
\begin{align*}
\tilde{C}_t + I_t &= \tilde{K}_t^{\alpha} N_t^{1 - \alpha} \\
e^{\Delta a_{t+1}} \tilde{K}_{t+1} &= (1 - \delta) \tilde{K}_t + \phi \left( \frac{I_t}{\tilde{K}_t} \right) \tilde{K}_t \\
\Delta a_t &= \mu (s_t^\mu) + \sigma (s_t^\sigma) \epsilon_t, \quad \epsilon_t \sim N(0, 1),
\end{align*}
$$

The numerical algorithm for computing this equilibrium is illustrated in the Appendix.

### 3.4 Asset Prices

The stochastic discount factor for the generalized recursive smooth ambiguity utility is given by (see Ju and Miao (2012) and Hayashi and Miao (2011))

$$
M_{t,t+1} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \left( \frac{1 - N_{t+1}}{1 - N_t} \right)^{\frac{1 - \psi}{\nu}} \left( \frac{V_{t+1}}{R_{t+1}} \right)^{\frac{1}{\psi} - \gamma} \left( \frac{\mathbb{E}_{z_{t+1}, \mu_{t+1}, \nu_{t+1}, \tau_{t+1}} \left[ V_{t+1}^{1 - \gamma} \right]}{\mathbb{R}_{t} (V_{t+1})} \right)^{\frac{1 - \gamma}{\lambda}}.
$$

The last multiplicative term in the SDF reflects pessimism due to ambiguity aversion. This pessimistic distortion to the SDF is endogenous and time-varying and makes the SDF more countercyclical. As a result, the model is able to create large equity premium and variance risk premium. Without ambiguity aversion, only long run consumption risks are priced. When risk aversion is low, the model generates too low equity premium and variance risk premium despite the presence of time-varying uncertainty.

The risk-free rate, $R_{f,t}$, is the reciprocal of the expectation of the pricing kernel:

$$
R_{f,t} = \frac{1}{\mathbb{E}_t [M_{t,t+1}]}.
$$

Following standard Q-theory arguments, Tobin’s $Q$ is given by

$$
q_t = \frac{1}{\phi' \left( \frac{I_t}{\tilde{K}_t} \right)}
$$
where $\phi'$ is the partial derivative of the adjustment cost function.

In RBC models, the firm’s payout in period $t$ is given by

$$D^*_t = Y_t - w_t N_t - I_t = \alpha Y_t - I_t,$$

where $w_t$ is the equilibrium wage and the second equality follows from the first order condition of labor: $w_t = \partial Y_t / \partial N_t = (1 - \alpha) A_t^{1-\alpha} K_t^{-\alpha}$. The return on capital (investment), $R^K_{t+1}$, is

$$R^K_{t+1} = \frac{1}{q_t} \left\{ q_{t+1} \left[ 1 - \delta K_{t+1}^{\alpha} \left( \frac{I_{t+1}}{K_{t+1}} \right) \right] + \alpha Y_{t+1} - I_{t+1} \right\}$$

In the absence of labor market frictions (for example, search frictions or wage rigidity), the model implies that the firm’s payout is countercyclical: investment is high in good times, resulting in a reduction in the firm’s payout, given that the model is calibrated to reproduce high investment volatility. The countercyclicality of the payout greatly reduces the riskiness of the equity claim and implies very low equity premium.

Because the stock market dividends only account for a small fraction of the payouts of production units (e.g., private equity, small businesses, and real estate), we follow Bansal and Yaron (2004), Ju and Miao (2012), Kaltenbrunner and Lochstoer (2010) and Liu and Miao (2015) and assume that aggregate dividends are defined as a levered claim to aggregate consumption. In particular, the dividend growth process is specified as containing a component proportional to consumption growth and an independent component,

$$\Delta d_{t+1} \equiv \ln \left( \frac{D_{t+1}}{D_t} \right) = \lambda \Delta c_{t+1} + g_d + \sigma_d \varepsilon_{d,t+1}$$

where $\varepsilon_{d,t+1}$ is an IID standard normal random variable and is independent of all other shocks in the model. The parameter $\lambda$ can be interpreted as the leverage ratio on expected consumption growth as in Abel (1999). The parameters $g_d$ and $\sigma_d$ are calibrated to match the first and second moments of dividend growth in the data. The specification implies that dividend growth is procyclical as opposed to being countercyclical in standard RBC models. This feature allows the model to better match equity premium and equity volatility in the data.

Stock returns, $R_{t+1}$, are defined by

$$R_{t+1} \equiv \frac{P_{t+1} + D_{t+1}}{P_t} = \frac{1 + P_{t+1}/D_{t+1}}{P_t/D_t} = \frac{D_{t+1}}{D_t}$$
and satisfy the Euler equation
\[ \mathbb{E}_t [M_{t,t+1} R_{t+1}] = 1 \]

To solve for the price-dividend ratio, we rewrite the Euler equation as
\[ \frac{P_t}{D_t} = \mathbb{E}_t \left[ M_{t,t+1} \left( 1 + \frac{P_{t+1}}{D_{t+1}} \right) \frac{D_{t+1}}{D_t} \right] \]

Using the state variables to express the price-dividend ratio and setting \( \frac{P_t}{D_t} = \xi \left( \tilde{K}_t, s_{t}^{\mu}, s_{t}^{\sigma} \right) \), the Euler equation becomes
\[ \xi \left( \tilde{K}_t, s_{t}^{\mu}, s_{t}^{\sigma} \right) = \mathbb{E}_t \left[ M_{t,t+1} \left( 1 + \xi \left( \tilde{K}_{t+1}, s_{t+1}^{\mu}, s_{t+1}^{\sigma} \right) \right) \exp (\Delta d_{t+1}) \right] \]

This functional equation can be solved by approximating the price-dividend ratio with Chebyshev polynomials in the state variables. The numerical method is explained in the Appendix.

How does ambiguity aversion alter the pricing of risky assets in equilibrium? To answer this question, we begin the analysis by decomposing the SDF as
\[ M_{t,t+1} = M_{t,t+1}^{EZ} M_{t,t+1}^{AA} \]

with
\[ M_{t,t+1}^{EZ} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \left( \frac{1 - N_{t+1}}{1 - N_t} \right)^{(1-\frac{1}{\psi})\nu} \left( \frac{V_{t+1}}{R(t,V_{t+1})} \right)^{\frac{1}{\psi} - \gamma} \]

and
\[ M_{t,t+1}^{AA} = \left( \frac{E_{z_{t+1},t} \left[ V_{t+1}^{1-\gamma} \right]}{R(t,V_{t+1})} \right)^{-(\eta-\gamma)} . \]

Let \( z = 1, 2, 3, \) and \( 4 \) correspond to, respectively, states \((\mu_l, \sigma_l), (\mu_h, \sigma_l), (\mu_l, \sigma_h)\) and \((\mu_h, \sigma_h)\). The Markov switching process (1) implies that the physical transition probabilities are
\[
\mathbb{P} = \begin{pmatrix}
(1 - p_{ll}^\mu p_{ll}^\sigma) & p_{ll}^\mu (1 - p_{ll}^\sigma) & (1 - p_{ll}^\mu) p_{ll}^\sigma & (1 - p_{ll}^\mu) (1 - p_{ll}^\sigma) \\
(1 - p_{lh}^\mu) p_{ll}^\sigma & p_{lh}^\mu p_{hh}^\sigma & (1 - p_{lh}^\mu) p_{lh}^\sigma & (1 - p_{lh}^\mu) p_{lh}^\sigma \\
(1 - p_{hh}^\mu) p_{ll}^\sigma & (1 - p_{hh}^\mu) (1 - p_{ll}^\sigma) & p_{hh}^\mu p_{ll}^\sigma & p_{hh}^\mu (1 - p_{ll}^\sigma) \\
(1 - p_{hh}^\mu) (1 - p_{hh}^\sigma) & (1 - p_{hh}^\mu) (1 - p_{hh}^\sigma) & p_{hh}^\mu (1 - p_{hh}^\sigma) & p_{hh}^\mu p_{hh}^\sigma
\end{pmatrix} .
\]

The Euler equation can be rewritten as
\[ 0 = \sum_{z_{t+1}} p(z_t, z_{t+1}) \mathbb{E}_{z_{t+1},t} \left[ M_{t,t+1} (R_{t+1} - R_{f,t}) \right] , \text{ for } z_t = \{1, 2, 3, 4\} \]
Applying the SDF decomposition, we obtain

\[ 0 = \sum_{z_{t+1}} p(z_t, z_{t+1}) M^{AA}_{z_{t+1}, t} E_{z_{t+1}, t} \left[ M^{EZ}_{t, t+1} (R_{t+1} - R_{f, t}) \right] \]

The Euler equation becomes

\[ 0 = \sum_{z_{t+1}} \hat{p}(z_t, z_{t+1}) E_{z_{t+1}, t} \left[ M^{EZ}_{t, t+1} (R_{t+1} - R_{f, t}) \right] \]

where \( \hat{p}(z_t, z_{t+1}) \) are distorted transition probabilities induced by ambiguity aversion

\[ \hat{p}(z_t, z_{t+1}) = \frac{p(z_t, z_{t+1}) M^{AA}_{z_{t+1}, t}}{\sum_{z_{t+1}} p(z_t, z_{t+1}) M^{AA}_{z_{t+1}, t}} \quad (5) \]

Thus, once the equilibrium allocation has been found, the economy is observationally equivalent to an economy with Epstein-Zin preferences and the time-varying distorted transition probabilities \( \hat{p}(z_t, z_{t+1}) \). It is important to note that the distortion induced by ambiguity aversion depends on the state of the economy and arises as an equilibrium outcome. Furthermore, although the physical transition probabilities imply independent mean and volatility regime switching, the distorted transition probabilities do not.

It can be shown that for any current state \( z_t \), the distorted transition probabilities “steer” toward future bad states but away from future good states: instead of fully trusting the physical transition probabilities, the ambiguity-averse agent believes that starting from the current state, the economy is more (less) likely to switch to bad (good) states in the next period. The quantitative analysis in the next section shows this result.

The size of the distortion is mainly determined by the degree of ambiguity aversion. Since the analysis above shows that the distortion is applied to the physical transition probabilities of the Markov chain, we can exploit this result to calibrate the ambiguity aversion parameter. In particular, we use detection-error probabilities to calibrate \( \eta \). The method of computing detection-error probabilities is developed by Anderson et al. (2003). Jahan-Parvar and Liu (2014) apply the method to smooth ambiguity preferences in a dynamic setting. Detection-error probabilities inform us the probability of making errors in distinguishing the reference model from the distorted model by means of comparing likelihoods under the two models. With a large value of \( \eta \), the magnitude of the distortion is significant, and thus the reference model (transition probabilities) looks very differently from the distorted model (transition probabilities). As a consequence, it will
be easy to distinguish the two models, resulting in a small detection-error probability. The closed-form formulas for the distorted transition probabilities make it feasible to simulate data from the distorted Markov switching process and evaluate the likelihood of simulated data. The Appendix contains the details of computing detection-error probabilities.

### 3.5 Variance risk premium

In the model, variance risk premium is defined as the difference between the risk-neutral and objective expectations of stock return variance

$$VRP_t = \frac{E_t \left[ M_{t,t+1} \sigma^2_{R,t+1} \right]}{E_t \left[ M_{t,t+1} \right]} - E_t \left[ \sigma^2_{R,t+1} \right]$$

where $\frac{M_{t,t+1}}{E_t[M_{t,t+1}]}$ characterizes the risk-neutral measure transformation, and $\sigma^2_{R,t}$ denotes the conditional variance of equity returns, i.e., $\sigma^2_{R,t} \equiv \text{Var}_t (R_{t+1})$.

Applying the SDF decomposition, we can rewrite the risk-neutral expectation of return variance in the following

$$E_t \left[ M_{t,t+1} \sigma^2_{R,t+1} \right] = \sum_{z_{t+1}} E_{z_{t+1},t} \left[ M_{t,t+1} \sigma^2_{R,t+1} \right] M^{AA}_{z_{t+1},t} p(z_t, z_{t+1})$$

where the term $E_{z_{t+1},t} \left[ M_{t,t+1} \sigma^2_{R,t+1} \right]$ can be further rewritten as

$$E_{z_{t+1},t} \left[ M_{t,t+1} \sigma^2_{R,t+1} \right] = E_{z_{t+1},t} \left[ M_{t,t+1} \right] E_{z_{t+1},t} \left[ \sigma^2_{R,t+1} \right] + \text{cov}_{z_{t+1},t} (M_{t,t+1}, \sigma^2_{R,t+1}).$$

It then follows that

$$\frac{E_t \left[ M_{t,t+1} \sigma^2_{R,t+1} \right]}{E_t \left[ M_{t,t+1} \right]} = \sum_{z_{t+1}} E_{z_{t+1},t} \left[ M_{t,t+1} \right] E_{z_{t+1},t} \left[ \sigma^2_{R,t+1} \right] M^{AA}_{z_{t+1},t} p(z_t, z_{t+1})$$

$$\sum_{z_{t+1}} E_{z_{t+1},t} \left[ M_{t,t+1} \sigma^2_{R,t+1} \right] M^{AA}_{z_{t+1},t} p(z_t, z_{t+1})$$

$$+ \sum_{z_{t+1}} \text{cov}_{z_{t+1},t} \left( M_{t,t+1}, \sigma^2_{R,t+1} \right) M^{AA}_{z_{t+1},t} p(z_t, z_{t+1}).$$

The objective expectation of the conditional variance can be alternatively written as

$$E_t \left[ \sigma^2_{R,t+1} \right] = \sum_{z_{t+1}} E_{z_{t+1},t} \left[ \sigma^2_{R,t+1} \right] p(z_t, z_{t+1}).$$
Define the risk-neutral transition probabilities by
\[
\tilde{p}(z_t, z_{t+1}) = \frac{E_{z_{t+1}, t} \left[ M_{t,t+1}^{EZ} \right] M_{z_{t+1}, t}^{AA} p(z_t, z_{t+1})}{\sum_{z_{t+1}} E_{z_{t+1}, t} \left[ M_{t,t+1}^{EZ} \right] M_{z_{t+1}, t}^{AA} p(z_t, z_{t+1})}
\]
where the term
\[
\frac{E_{z_{t+1}, t} \left[ M_{t,t+1}^{EZ} \right] M_{z_{t+1}, t}^{AA}}{\sum_{z_{t+1}} E_{z_{t+1}, t} \left[ M_{t,t+1}^{EZ} \right] M_{z_{t+1}, t}^{AA}}
\]
is the Radon-Nikodym derivative introduced by the risk-neutral measure. Finally, the VRP can be decomposed in the following
\[
VRP_t = \sum_{z_{t+1}} E_{z_{t+1}, t} \left[ \sigma_{R,t+1}^2 \right] \left[ \tilde{p}(z_t, z_{t+1}) - p(z_t, z_{t+1}) \right] + \sum_{z_{t+1}} \frac{\text{cov}_{z_{t+1}, t} \left( M_{t,t+1}^{EZ}, \sigma_{R,t+1}^2 \right)}{E_{z_{t+1}, t} \left[ M_{t,t+1}^{EZ} \right]} \tilde{p}(z_t, z_{t+1})
\]
(6)

In the first term, \([\tilde{p}(z_t, z_{t+1}) - p(z_t, z_{t+1})]\) is the difference between the risk-neutral and objective transition probabilities, and \(E_{z_{t+1}, t} \left[ \sigma_{R,t+1}^2 \right]\) is the expected conditional variance in state \(z_{t+1}\). In the second term, \(\text{cov}_{z_{t+1}, t} \left( M_{t,t+1}^{EZ}, \sigma_{R,t+1}^2 \right)\) is the conditional covariance between the SDF for Epstein-Zin recursive utility and the conditional variance.

4 Calibration

To calibrate the production economy to the historical data, we compute unconditional moments of quantities and asset returns for the period 1952:Q1—2012:Q4. The sample statistics include the standard deviations and correlations of investment, consumption, and output growth, the first autocorrelation in consumption growth, the first and second moments of the risk-free rate and equity returns, and moments of VRP. Nominal returns are deflated by the CPI data from FRED at St.Louis. The model is calibrated at a quarterly frequency. Because the model is nonlinear, no analytical solutions are available. We solve the model using numerical methods and then simulate the model at a quarterly frequency. The quantitative results are based on 10,000 simulations.

4.1 Parameter choice

The capital share is set at \(\alpha = 0.36\) and the depreciation rate at \(\delta = 0.02\), following the RBC literature. The leisure preference parameter \(\nu\) in the felicity function is set at 2 such that the long run mean proportion of labor hours is 30%. We follow the production-based long run risk literature
(Croce (2010) and Liu and Miao (2015)) and fix the EIS parameter at a value greater than 1, \( \psi = 1.5 \). There is still no consensus regarding the value of the EIS parameter. Some papers find that the estimate of \( \psi \) is close to 0 (Hall (1988), Campbell and Mankiw (1989), and Ludvigson (1999)), while others find that the estimate is large and typically above 1 (Vissing-Jorgensen (2002) and Attanasio and Vissing-Jorgensen (2003)). We also present results of the sensitivity analysis by varying the EIS parameter. The coefficient of risk aversion is set at a low value, \( \gamma = 1.5 \), in the benchmark calibration. The capital adjustment costs parameter \( \xi \) is set to match the ratio between the volatility of investment growth and that of consumption growth. The subjective discount parameter \( \beta \) is chosen to imply a low mean risk-free rate.

[Insert Table 2 about here]

The financial leverage parameter \( \lambda \) takes the value \( \lambda = 2.5 \), close to the values adopted by Abel (1999) and Bansal and Yaron (2004). The quarterly standard deviation of dividends growth is set at 6% such that the benchmark model can match the equity volatility in the data. This value is close to the values considered in other calibration studies (Bansal and Yaron (2004) and Ju and Miao (2012)). The standard deviation of the independent shock in the dividend growth dynamics, \( \sigma_d \), is set to match the volatility of dividend growth, which implies \( \sigma_d = 0.057 \), given that the model implied volatility of consumption growth is about 0.65 percent per quarter. Following Bansal and Yaron (2004), \( g_d \) is chosen such that the average rate of dividends growth is equal to that of consumption growth, which delivers \( g_d = -0.38\% \). The calibration implies that the contemporaneous correlation between consumption growth and dividends growth is about 0.28, close to the value considered by Kaltenbrunner and Lochstoer (2010).

In the benchmark model (AA1), the ambiguity aversion parameter is set at \( \eta = 30 \), to match the mean equity premium in the data. The detection-error probability associated with \( \eta = 30 \) is 28.39%. This finding suggests that the implied distorted Markov switching process is not “far” from the reference process by means of relative entropy and that the probability of making mistakes in distinguishing the two processes is still high enough. Jahan-Parvar and Liu (2014) calibrate a value of \( \eta \) around 19, based on the long sample including the 1930s Great Recession. Chen et al. (2014) consider a much higher value of \( \eta \) in analyzing dynamic portfolio choice.
4.2 Impulse responses

To examine the impacts of exogenous shocks to productivity growth, we consider three independent cases: an innovation shock to the growth rate and two regime shifts in its conditional mean and volatility. We suppose that the economy stays in state \((\mu_h, \sigma_l)\) for a long time without the impact of innovation shocks. The aggregate capital stays at its steady-state level. Assuming that each of the three shocks hits the economy in the third period and only lasts for that period, We investigate the impulse responses of quantities, the equity return, equity volatility and the risk-neutral expected return variance.

Figure 6 plots the impulse responses of variables in interest to a negative innovation shock with the size \(2\sigma_l\). The consumption and investment series are detrended in the plot. Because the impact of the innovation shock on the productivity level is permanent, both investment and consumption decrease temporarily and then move to their new steady-state levels. The decline in investment implies that expected future consumption growth is low, generating the long run risk effect on the SDF. Hours worked also decreases and then slowly reverts back to its original steady-state. Altogether, these effects lead to an increase in marginal utility of the representative agent and thus an increase in the SDF. However, since the productivity growth regime remains unchanged, ambiguity aversion has little impact on the response of the SDF. In addition, the price-dividend ratio and realized return drop mildly as expected productivity growth remains the same. The lower price-dividend ratio leads to rises in conditional equity premium and equity volatility, though their responses are not substantial. Because both the SDF and conditional equity volatility are not significantly responsive to the innovation shock, the risk-neutral variance does not change much, as shown in the last plot in Figure 6. These results also imply that the VRP is not sensitive to the innovation shock to productivity growth.

[Insert Figure 6 about here]

Figure 7 displays the impulse responses results for mean switching in productivity growth. The productivity growth regime shifts to the state \((\mu_l, \sigma_l)\). The quantity dynamics exhibit a similar pattern to the previous case of the innovation shock, but the responses of financial variables are much more significant. Because the mean growth rate shifts to the bad state, the ambiguity-averse agent slants his belief about the growth rate regimes in a pessimistic way. The last multiplicative
term in the SDF plays an important role and increases the SDF substantially. Again, due to the 
induced pessimism, the price-dividend ratio falls by a large amount, resulting in significant increases 
in the conditional equity premium and equity volatility. Because both the SDF and conditional 
equity volatility move in the positive direction and in large magnitude, the risk neutral variance 
rises by a large amount. Thus, the mean switching in productivity drives countercyclical variation 
in the risk neutral variance. These results provide direct support to the finding in Section 2 that 
financial uncertainty proxied by the risk neutral variance is negatively priced and carries a positive 
risk premium.

[Insert Figure 7 about here]

Turning to the impacts of volatility regime switching, Figure 8 plots the impulse responses 
when the economy switches to the high volatility state \((\mu_h, \sigma_h)\). Because the regime shift is on the 
second moment of the growth rate, the productivity level does not change. Hours worked fall on 
impact. Due to the fact that consumption and leisure are complements, consumption increases but 
investment decreases. As a consequence, the short-run effect caused by higher current consumption 
and the long-run effect by lower expected future consumption growth affect the SDF in opposite 
directions. Although ambiguity aversion also has a positive impact on the response of the SDF, 
the net effect on the SDF is still insignificant. Consistent with Liu and Miao (2015) and Bansal 
et al. (2013), the plot shows that the volatility shock has negative market price of risk in that the 
realized equity return drops on impact but the conditional equity premium rises. In addition, as 
the economy falls in the high volatility state, the state change implies that both the conditional 
equity volatility and risk neutral variance increase by larger amounts than in the case with the 
innovation shock (6).

[Insert Figure 8 about here]

4.3 Quantitative results

Table 3 summarizes the annualized moments for macroeconomic quantities and asset returns. The 
following observations are in order: (1) the volatility of consumption growth is low \((\sigma_{\Delta c} = 0.96)\), (2) 
the volatility of investment growth is much higher \((\sigma_{\Delta i} = 4.5)\), (3) comovements among aggregate 
quantities, (4) the risk-free rate is low and smooth, and (5) equity premium and equity volatility 
are high \((\mathbb{E}(R - R_f) = 6.35\%\) and \(\sigma(R - R_f) = 16.85\%)\).
4.3.1 Model comparison

We first present simulation results for the benchmark model with ambiguity aversion and time-varying conditional mean and volatility of productivity growth and then compare results with models under alternative assumptions. The benchmark model AA1 can match the moments of quantities well, despite that the model implied volatility of consumption growth is moderately higher than in the data. Because adjustment costs are assumed to be low (\( \xi = 3 \)) and the EIS is high (\( \psi = 1.5 \)), investment growth is volatile in the model (\( \sigma_{\Delta i} = 5.13 \)). A high EIS increases the intertemporal substitution effect and makes savings more appealing in high productivity states. With endogenous labor hours, the correlation between consumption growth and investment growth (\( \rho(\Delta c, \Delta i) = 0.66 \)) is closer to the data than in models with exogenous labor choice, for example, see Kaltenbrunner and Lochstoer (2010) and Liu and Miao (2015). Due to persistence in productivity growth regimes, the benchmark model can also reproduce persistence in consumption growth observed in the data, though the magnitude cannot match the data.

Because the EIS parameter value is high, the intertemporal substitution effect implies a low volatility of the risk-free rate. With low risk aversion, the benchmark model can closely match the equity premium and equity volatility in the data. As shown in the impulse responses analysis, each of the three independent shocks to productivity growth generates negative correlation between the equity return and SDF. Moreover, ambiguity aversion greatly increases the variation in the SDF and implies high market price of risk \( \sigma(M)/E(M) = 0.83 \). According to the conditional version of the Euler equation (see Cochrane (2005)):

\[
E_t(R_{t+1}) - R_{f,t} = \frac{\sigma_t(M_{t+1})}{E_t(M_{t+1})} \sigma_t(R_{t+1}) \rho_t(M_{t+1}, R_{t+1}) ,
\]

the model implies a high equity premium of about 7 percent per year.

Remarkably, the benchmark model produces a high VRP of 5.26 in squared percentage terms, which is about half of the average VRP in the data (12.08 for the full sample period). Further increasing the ambiguity aversion parameter can match the average VRP in the data but will generate a mean equity premium close to 10 percent per year. The standard deviation of the VRP implied by the model is 5.78, which is still far below that in the data (34.95 for the full sample period and 17.34 for the pre-crisis period). Consistent with the data, the model generates a non-Gaussian distribution of VRP, with a skewness measure of 4.12 and an excess kurtosis of 23.66.
The magnitude of kurtosis implied by the benchmark model is close to that in the data (35.31 for the full sample period and 10.82 for the pre-crisis period), but the model implied positive skewness is at odds with the full sample skewness, due to the substantial realized variance during the crisis period.

The model can also generate implications on the relations between financial uncertainty proxied by the risk neutral variance and key macroeconomic and financial variables. The benchmark model generates correlograms between the risk neutral variance and aggregate quantities in line with the data. Figure 4 shows that the current risk neutral variance is negatively correlated with output, consumption and investment at different leads and lags. In addition, the model replicates the lead-lag relation between the risk neutral variance and equity return quite well: low past returns lead to high current risk neutral variance, which subsequently leads to high future returns. The magnitude of the correlations implied by the model is even close to that in the data. These results are mainly driven by Markov switching productivity growth. In the high mean growth regime, the risk neutral variance is low, but consumption and investment tend to be high. Due to persistence of growth regimes, quantities will remain at high levels in future periods. Additionally, because the high mean growth regime represents a low uncertainty state, conditional equity premium is small, which leads to low future returns. The opposite scenario occurs when productivity growth shifts to the low mean regime.

Given that the model also features stochastic volatility, we next examine the relations between the risk neutral variance and volatilities of aggregate quantities and returns. Figure 5 shows that the model can reproduce well the patterns observed in the data. The risk neutral variance exhibits positive correlations with volatilities of output growth, investment growth, consumption growth and the equity return at different leads and lags. The impulse responses analysis with respect to the volatility regime switching suggests that the high productivity growth volatility regime is associated with high conditional equity volatility and high risk neutral variance. Naturally, aggregate quantities also become more volatile when productivity growth has more volatility.

For comparison, model EZ1 assumes that the representative agent has Epstein-Zin recursive utility with \( \gamma = 1.5 \). Table 3 shows that the quantity dynamics are not affected much, except that the volatility of investment (consumption) growth is moderately lower (higher) than in the benchmark model. Without ambiguity aversion, the precautionary savings motive is dampened and the investment level becomes lower. Investment becomes less responsive to productivity fluctuations,
and thus consumption absorbs a larger fraction of the variation in productivity. On the other hand, the impact on financial moments is significant. In model EZ1, the risk-free rate is excessively high while both equity premium and the VRP are close to 0. The model implied market price of risk is also close to 0. These results are due to the fact that in model EZ1, the business cycle fluctuations are unable to create large variations in the SDF and conditional volatility of equity returns.

An alternative model is model EZ2, where the agent has Epstein-Zin preferences but a much higher risk aversion parameter is considered ($\gamma = 16.5$). Similar to the benchmark model, this model can explain key features of aggregate quantities, the risk-free rate and equity premium well. However, the model implied VRP is too low in comparison with the actual data. This result indicates that the implication on the VRP of time-varying pessimism induced by ambiguity version cannot be simply achieved by simply increasing risk aversion. Moreover, the required risk aversion $\gamma = 16.5$ exceeds the commonly considered reasonable range $\gamma \in [1, 10]$ (Mehra and Prescott (1985)).

Model AA differs from the benchmark model by assuming that the innovation shock to productivity growth has constant volatility. The volatility of the innovation shock is set at the steady-state level implied by the Markov switching process. Compared to the benchmark model, model AA generates lower volatilities of growth rates of consumption, investment and output, a lower risk-free rate, and lower equity premium and variance risk premium. The impact of the productivity volatility risk on equity premium is moderate. This is in contrast to Liu and Miao (2015), where generalized disappointment aversion makes the volatility risk priced significantly. It is also worth noting that because model AA has no stochastic volatility, it cannot reproduce the empirical correlations between the VRP and time-varying volatilities of macroeconomic fundamentals.

The last model examined is model IID with Epstein-Zin preferences and IID productivity growth, where the mean and volatility are set at their corresponding steady-state levels implied by the Markov switching process. The risk aversion parameter value is chosen to produce an equity premium comparable to that in the benchmark model. This implies $\gamma = 31$, which is far greater than the plausible range of risk aversion. With constant mean and volatility of productivity growth, investment volatility becomes lower. In addition, the model loses some persistence in consumption growth without persistent productivity growth regimes. High risk aversion can help the model match the mean risk-free rate and equity premium. The implied market price of risk is also comparable to that in model AA1. However, the variance risk premium in the model
is too small ($\mathbb{E}(VRP) = 0.61$) because the model lacks mechanism to generate high conditional equity volatility and a pessimistic risk-neutral density over states. It is interesting to note that even this model can produce a non-Gaussian distribution of the VRP with positive skewness and high kurtosis similar to the data.

Empirical literature finds that the market VRP has predictability for stock returns, for example, see Bollerslev et al. (2009), Drechsler and Yaron (2011), Zhou (2010) and Feunou et al. (2015). The predictability is significant for short to medium horizons of returns, typically within a year, while results are mixed for forecasting long horizon returns. Table 4 presents the predictability results simulated from the benchmark model AA1 and model EZ1, where the log dividend yield and VRP are considered as predictors separately, and the horizon of equity returns ranges from a quarter to five years. For each simulated sample, the predictive regressions are estimated using the standard ordinary least square method. The results reported are based on averaging over 10,000 simulated samples. Model AA1 successfully reproduces the predictability by the dividend yield: the current dividend yield positively forecasts future returns, and the average $R^2$ of the predictive regression is increasing in the return horizon. The model also generates predictability by the VRP: a high current VRP forecasts high future returns. The average $R^2$ is 3.5 percent at a quarter horizon. Since the VRP is the only predictor in the regression, the $R^2$ obtained indicates significantly predictability by the VRP. By contrast, model EZ1 cannot generate significant predictability patterns, given very low simulated $R^2$'s. Because the model features long run risks, the predictability increases over longer horizons than a quarter. Thus, the model cannot replicate the empirical pattern that the predictability by the VRP has a hump-shape that peaks at one quarter horizon (Bollerslev et al. (2009)).

4.3.2 Conditional moments and the VRP decomposition

Table 5 presents conditional moments for models AA1, EZ1 and $\overline{\overline{A}}\overline{A}$ in the four states $(\mu_l, \sigma_l)$, $(\mu_l, \sigma_h)$, $(\mu_h, \sigma_l)$ and $(\mu_h, \sigma_h)$. In the benchmark model, financial moments differ most significantly across the two mean growth regimes. Suppose that the current volatility state is fixed at $\sigma_h$. Because low investment in the low mean growth regime leads to low expected consumption growth, the conditional risk-free rate in this regime is 0.22 percent per year, lower than that (1.73 percent) in the high mean growth regime. The conditional equity premium, equity volatility and variance risk premium in state $(\mu_l, \sigma_h)$ are much higher their counterparts in state $(\mu_h, \sigma_h)$. This result is
similar when the current volatility state is $\sigma_l$. The low mean growth regime is relatively transitory. Furthermore, ambiguity aversion reinforce the agent’s concern about this unfavorable and uncertain state, which raises the market price of risk in this state ($\sigma(M)/\mathbb{E}(M) = 1.03$). Thus, the agent demands a high risk premium. In addition, the conditional volatility of investment growth is high in this state ($\sigma_{\Delta i} = 6.74\%$), making future consumption and dividends more volatile. This effect greatly raises the conditional equity volatility. Together with strong pessimism, the low mean growth state implies a very high variance risk premium. These findings suggest that the conditional financial moments in the low mean growth regimes contribute to the success of the benchmark model in matching unconditional moments.

Comparing conditional moment across different volatility regimes ($\mu_h, \sigma_l$) and ($\mu_h, \sigma_h$), we observe that conditional volatilities of consumption growth and investment growth are significantly higher in the high productivity volatility regime. The conditional equity volatility is also higher in this state. However, because the conditional market price of risk remains the same across the two regimes, the conditional equity premium does not change much and the conditional VRP is modestly higher in state ($\mu_h, \sigma_h$). This result also indicates the positive correlation between the VRP and conditional volatilities of macroeconomic fundamentals. By contrast, Panel B in Table 5 also presents conditional moments of model EZ1. The conditional macroeconomic moments display a similar pattern to results in Panel A. However, without strong pessimism induced by ambiguity aversion, the conditional market price of risk, equity premium, equity volatility and variance risk premium differ only slightly across the low mean and high mean growth regimes. In addition, the conditional VRP exhibits little variation across volatility regimes, leading to weak correlations between the VRP and volatilities of quantities.

To understand the VRP dynamics better, We perform the VRP decomposition for the benchmark model as specified in (6). Table 6 summarizes the decomposition results. Panel A indicates that the first component in (6) accounts for almost all of the mean VRP in the data. We then compute the average of the component $\mathbb{E}_{z_{t+1}} \left[ \hat{P}(z_t, z_{t+1}) \right] \left[ \sigma_{R,t+1}^2 \right] [\hat{P}(z_t, z_{t+1}) - p(z_t, z_{t+1})]$ in each state $z_{t+1}$, where the average is taken over period-$t$ states. The decomposition shows that if the next period state ($z_{t+1}$) features high mean productivity growth, the component is negative, whereas it becomes positive and large for $z_{t+1}$ being low mean growth regimes. The intuition is the following: in expecting the next period growth state, the ambiguity-averse agent optimally alters the risk-neutral transition probabilities $\hat{p}(z_t, z_{t+1})$ such that the high mean growth regimes receive smaller weight.
while the low mean growth regimes receive larger weight, due to the induced pessimism. This explains the signs of the component observed in Panel A of Table 6. As the conditionally expected variance \( \mathbb{E}_{z_{t+1}, t}\left[\sigma^2_{R,t+1}\right] \) is far greater in regimes featuring low mean growth than in regimes with high mean growth, the risk-neutral expected conditional variance has been largely raised.

This intuition can be confirmed by results presented in Panel B, Table 6, where \( \mathbb{E}_{z_{t+1}, t}\left[\sigma^2_{R,t+1}\right] \), \( \tilde{p}(z_t, z_{t+1}) \) and \( p(z_t, z_{t+1}) \) are shown in each pair of state transition \((z_t, z_{t+1})\). The objective transition probabilities \( p(z_t, z_{t+1}) \) are obtained from the estimates in Table 1. The risk-neutral transition probabilities \( \tilde{p}(z_t, z_{t+1}) \) are computed as an equilibrium outcome. Comparing \( \tilde{p}(z_t, z_{t+1}) \) and \( p(z_t, z_{t+1}) \), it is obvious that the risk-neutral measure represents a dramatically pessimistic belief about the economy. For \( z_{t+1} \) being states \((\mu_l, \sigma_l)\) and \((\mu_l, \sigma_h)\), the risk-neutral probabilities exceed the objective probabilities, in some cases by a large amount. For instance, the risk-neutral probability for the transition from \( z_t = (\mu_l, \sigma_h) \) to \( z_{t+1} = (\mu_l, \sigma_h) \) is 0.821, whereas the objective probability is 0.573; in addition, the associated expected conditional variance is 110.31 in squared percentage terms. For \( z_{t+1} \) being states \((\mu_h, \sigma_l)\) and \((\mu_h, \sigma_h)\), \( p(z_t, z_{t+1}) \) is higher than \( \tilde{p}(z_t, z_{t+1}) \), though the expected conditional variance is much smaller in these states.

### 4.3.3 Sensitivity analysis

We perform several sensitivity analyses to study the impacts of model parameter values on macroeconomic and financial moments. When a different parameter value is considered, everything else remains the same as in the benchmark model. The results are summarized in Table 7.

The results for three alternative values of \( \eta \) indicate that the impact on consumption volatility and investment volatility is small. Backus et al. (2015) has a similar finding. The second observation is that the market price of risk and the first and second moments of equity premium and the VRP are significantly higher for higher values of \( \eta \). More ambiguity aversion imputes additional time-varying pessimism and thus raises risk premium in both equities and the conditional variance. For \( \eta = 40 \), the model can match the mean VRP in the data, but the implied equity premium is about 14 percent, which is much higher than the historical data. Changing risk aversion \( (\gamma) \) has little impact on consumption volatility and investment volatility. In a risk-sensitive framework, Tallarini (2000) concludes that the business cycle moments are not affected by changing risk aversion. Risk aversion does have a large impact on equity premium. For \( \gamma = 10 \), the model implied equity premium is about 10 percent. However, its impact on the VRP moments is of second order, unlike
ambiguity aversion.

The effect of the EIS parameter on volatilities of quantities is large. A low EIS implies strong aversion toward intertemporal substitution. In this case, investment is not sufficiently responsive to productivity shocks, which are, however, mostly absorbed by consumption choice. This effect generates more volatile consumption growth and less volatile investment growth. The EIS also affects financial moments through the channel of long run risks. The agent with a high EIS prefers earlier resolution of uncertainty. Thus, long run risks in productivity growth are significantly priced and carry positive risk premiums, leading to large equity premium and variance risk premium. If the EIS is small ($\psi = 0.2 < 1/\gamma$), the agent prefers later solution of uncertainty. In this model, both the equity risk premium and variance risk premium are very small. As is well known in the RBC literature, capital adjustment costs have a large impact on quantities. In the model, low values of $\xi$ mean high adjustment costs. Because consumption growth becomes more volatile for a low value of $\xi$, the model implies higher equity premium, equity volatility and variance risk premium. However, the magnitude of changes is modest.

5 Conclusion

We have studied a production-based asset pricing model with time varying uncertainty and ambiguity aversion. The productivity growth rate follows a Markov switching process with stochastic conditional mean and volatility. The representative agent is assumed to have generalized recursive smooth ambiguity preferences. Under ambiguity aversion, the objective transition probabilities of productivity growth states are distorted in an endogenous and pessimistic way. This time-varying pessimistic distortion in the agent’s beliefs has important asset pricing implications especially for the variance risk premium. With a low relative risk aversion parameter and a high elasticity of intertemporal substitution, the model can reproduce salient features of macroeconomic quantities and asset returns in the data. Because ambiguity aversion optimally alters the risk-neutral probabilities such that high conditional volatility states receive more weight, the model can generate a sizable variance risk premium. The model implies that financial uncertainty proxied by the risk neutral variance is negatively correlated with aggregate quantities but positively correlated with their volatilities. In addition, the model generates positive correlations of the risk neutral variance with equity volatility and future equity returns. These model implications are fully consistent with
the observations in the data.
6 Appendix

6.1 Numerical algorithm

6.1.1 Solving the equilibrium

Because the model is nonlinear, the standard perturbation method cannot be applied to solve the model. We use the value function iteration method to solve the model numerically. The social planner’s problem is

\[
\tilde{J}_t = \max_{\tilde{C}_t, \tilde{I}_t, N_t} \left[ (1 - \beta) \left( \tilde{C}_t (1 - N_t) \right)^{1 - \rho} + \beta \left( \mathbb{E}_{z_{t+1}} \left[ \left( \left( \exp (\Delta a_{t+1}) \tilde{J}_{t+1} \right)^{1 - \gamma} \right)^{1 - \frac{1}{1 - \eta}} \right] \right)^{1 - \frac{1}{1 - \eta}} \right]^{1 - \frac{1}{1 - \rho}}
\]

subject to

\[
\begin{align*}
\tilde{C}_t + \tilde{I}_t &= \tilde{K}_t^\alpha N_t^{1 - \alpha} \\
\exp(\Delta a_{t+1}) \tilde{K}_{t+1} &= \left( 1 - \delta \right) \tilde{K}_t + \phi \left( \frac{\tilde{I}_t}{\tilde{K}_t} \right) \tilde{K}_t e^{(1 - s_{t+1}^\mu) \omega_{t+1}} \\
\Delta a_t &= \mu \left( s_t^\mu \right) + \sigma \left( s_t^\sigma \right) \epsilon_t, \quad \epsilon_t \sim N(0, 1)
\end{align*}
\]

The numerical algorithm proceeds as follows:

1. We compute the detrended capital \( \tilde{K}_{ss} \) in the deterministic steady state, assuming that the productivity growth rate is constant and equal to the steady-state level. The state space for \( \tilde{K} \) is set at \( \left[ \tilde{K}_{min}, \tilde{K}_{max} \right] = \left[ 0.2 \tilde{K}_{ss}, 2.0 \tilde{K}_{ss} \right] \). The capital grid has \( n_k \) grid points on this interval. The grid points are determined from Chebyshev zeros. We use \( n_k = 100 \) in the numerical computation. Further increasing \( n_k \) does not lead to significantly different results.

2. We pick a grid for \( \tilde{I} \). That is, for each capital grid point, We construct equidistant points for \( \tilde{I} \) bounded between 0 and \( \tilde{K}_t^\alpha \). We use 100 points for the \( \tilde{I} \) grid.

3. Compute for each \((\tilde{K}, \tilde{I})\) in the grid the value \( N(\tilde{K}, \tilde{I}) \) that solves

\[
\max_N \left( \tilde{K}_t^\alpha N^{1 - \alpha} - \tilde{I} \right) \left( 1 - N \right)^\nu
\]

For each \( \tilde{K} \) point, an interpolation method is applied to interpolate \( N(\tilde{K}, \tilde{I}) \) onto the grid for \( \tilde{I} \) and to obtain the interpolation coefficients.

4. For numerical integration with respect to \( \Delta a \), We use the procedure described in the Appendix
of Rouwenhorst (1995) to discretize the innovation shock with 9 grid points. Alternatively, one can use quadrature methods.

5. Our goal is to compute \( \tilde{J}(\tilde{K},z,s) \) on the grid \([\tilde{K}_{\text{min}}, \tilde{K}_{\text{max}}] \times s^\mu \times s^\sigma \), where \( s^\mu \) and \( s^\sigma \) are two binary state variables. For each state \( \{s^\mu, s^\sigma\} \), the value function \( \tilde{J}(\tilde{K}, s^\mu, s^\sigma) \) is approximated by Chebyshev polynomials in \( \tilde{K} \).

6. Denote the current period value function and the next period value function by

\[
\tilde{J} = \tilde{J}(\tilde{K}, s^\mu, s^\sigma)
\]

\[
\tilde{J}' = \tilde{J}(\tilde{K}', (s^\mu)', (s^\sigma)')
\]

Given the current state \( (\tilde{K}, s^\mu, s^\sigma) \), a numerical optimizer is used to solve the Bellman equation

\[
\tilde{J} = \max_{I,N} \left[ (1 - \beta)\tilde{U}^{1-\rho} + \beta \left( \mathbb{E}_{zt+1} \left( \left( \mathbb{E}_{zt+1,t} \left( \exp(\Delta a_{t+1}) \tilde{J}' \right)^{1-\gamma} \right)^{1-\eta} \right)^{1-\sigma} \right)^{1-\sigma} \right]
\]

where

\[
\tilde{U} = (\tilde{K}^\alpha N^{1-\alpha} - \tilde{I}) (1 - N)^\nu
\]

For \( \tilde{K}' \) outside the grid points, Chebyshev approximation is used to compute the value function. For a given investment policy \( \tilde{I} \), the B-spline method is used to compute the corresponding optimal \( N \).

7. The updated value function \( \tilde{J}^* \) on each grid in the state space after an iteration. The algorithm then returns to the previous step. The stopping rule is that the new value function and the old value function satisfies \( |\tilde{J}^* - \tilde{J}| / |\tilde{J}| < 10^{-12} \).

6.1.2 Solving asset prices

The pricing kernel can be rewritten as

\[
M_{t,t+1} = \beta \left( \frac{\tilde{C}_{t+1} e^{\Delta a_{t+1}}}{\tilde{C}_t} \right)^{-\rho} \left( \frac{1 - N_{t+1}}{1 - N_t} \right)^{(1-\rho)\nu} \left( \frac{\tilde{J}_{t+1} e^{\Delta a_{t+1}}}{\mathcal{R}_t} \right)^{\rho-\gamma} \left( \frac{\mathbb{E}_{zt+1,t} \left( \tilde{J}_{t+1}^{1-\gamma} e^{(1-\gamma)\Delta a_{t+1}} \right)^{1-\sigma}}{\mathcal{R}_t} \right)^{-(\eta-\gamma)}
\]
To solve for the equilibrium price-dividend ratio from the Euler equation, we use the following Chebyshev approximation

\[ \xi \{ s_t^\mu, s_t^\sigma \} (\tilde{K}) \simeq p \sum_{j=0}^{p} \theta \{ s_{t+1}^\mu, s_{t+1}^\sigma \} T_j \left( y \left( \tilde{K} \right) \right) \]

\( p \) is the order of Chebyshev polynomials, \( T_j \) \( j = 0, ..., p \) are Chebyshev polynomials, and \( y \left( \tilde{K} \right) \) maps the state variable \( \tilde{K} \) onto the interval \([-1, 1]\). Using a nonlinear equation solver to solve the Euler equation

\[ \xi \{ s_t^\mu, s_t^\sigma \} (\tilde{K}_t) = E_t \left[ M_{t,t+1} \left( 1 + \xi \{ s_{t+1}^\mu, s_{t+1}^\sigma \} (\tilde{K}_{t+1}) \right) \exp (\Delta d_{t+1}) \right] \]

We can obtain the coefficients \( \theta \{ s_t^\mu, s_t^\sigma \}, j, j = 0, ..., p \).

### 6.2 Detection-error probabilities

We use simulations to approximate detection-error probabilities. Given the distorted transition probabilities derived in (5), it is feasible to simulate artificial data from both the reference Markov switching model and the distorted model. The numerical algorithm is described in the following steps.

1. Simulate \( \{ \Delta a_t \}_{t=1}^T \) under the reference model with the objective transition probabilities.

2. Compute the log likelihood function under the reference model:

\[
\ln L_T^r = \sum_{t=1}^T \ln \left\{ \sum_{z_t=1}^4 f (\Delta a_t | z_t) \Pr (z_t | \Omega_{t-1}) \right\}
\]

where \( \pi_{t-1} = \Pr (z_t = 1 | \Omega_{t-1}) \) are filtered probabilities implied by the Markov switching model.

3. Compute the log likelihood function under the distorted model:

\[
\ln L_T^d = \sum_{t=1}^T \ln \left\{ \sum_{z_t=1}^4 f (\Delta a_t | z_t) \Pr (z_t | \Omega_{t-1}) \right\}
\]

where \( \Pr (z_t | \Omega_{t-1}) \) are the filtered probabilities implied by the Markov switching model with the derived distorted transition probabilities. The fraction of simulations for which \( \ln \left( \frac{L_T^d}{L_T^r} \right) > 0 \) approximates the probability that the distorted model generated the data, while the data are actually generated by the reference model. This fraction is denoted by \( p_r \).
4. Perform a symmetrical computation and start by simulating \( \{\Delta a_t\}_{t=1}^T \) from the distorted model. The fraction of simulations for which \( \ln \left( \frac{L_T}{L_{T-1}} \right) > 0 \) fraction approximates the probability that the reference model generated the data when the artificial data are actually drawn from the distorted model. This fraction is denoted by \( p_d \).

Following Anderson et al. (2003), we assume the same prior on the reference and the distorted models, the detection-error probability is defined by

\[
p(\eta) = \frac{1}{2} (p_r + p_d).
\]

The length of the simulation is set at \( T = 200 \) quarters.

### 6.3 Data construction

- **Productivity level \( (A_t) \)** In constructing data on the level of aggregate productivity, we follow the methodology adopted by Stock and Watson (1999) and use data from the National Income and Product Accounts (NIPA). Quarterly data on Solow residuals are constructed from nonfarm gross domestic product (GDP) (Table 1.3.6, line 3), quarterly capital stock, and labor (hours of nonfarm employees). Quarterly capital values are constructed from interpolating annual non-residential capital stock (Fixed Assets Table 1.1, line 4) deflated by the price index for non-residential investment (Table 1.1.4, line 9) using quarterly non-residential fixed investment (Table 1.1.5, line 9) deflated by the corresponding price deflator. The capital share \( \alpha \) is assumed to be 0.36. The Solow residuals are rescaled by \( 1 - \alpha \) to obtain labor-augmenting technology level \( A_t \).
This table reports the maximum likelihood estimates of parameters in the Markov-switching model (1). Data for estimation are quarterly total factor productivity growth rates from 1952:Q1 to 2012:Q4. The estimates are obtained using the EM algorithm developed by Hamilton (1990). Standard errors are reported in parentheses.

<table>
<thead>
<tr>
<th>$\mu_l$</th>
<th>$\mu_h$</th>
<th>$\sigma_l$</th>
<th>$\sigma_h$</th>
<th>$p_{ll}^\mu$</th>
<th>$p_{hh}^\mu$</th>
<th>$p_{ll}^\sigma$</th>
<th>$p_{hh}^\sigma$</th>
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<td>0.755</td>
<td>1.816</td>
<td>0.586</td>
<td>0.973</td>
<td>0.967</td>
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<td>(0.096)</td>
<td>(0.161)</td>
<td>(0.159)</td>
<td>(0.318)</td>
<td>(0.034)</td>
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Table 2: **Benchmark Parameter Choices**

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<th>Parameter</th>
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<tr>
<td>$\beta$</td>
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<td>$\gamma$</td>
<td>Relative risk aversion</td>
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<td>$\psi$</td>
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<td>$\eta$</td>
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<td>$\nu$</td>
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<td>$\alpha$</td>
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<td>$\delta$</td>
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</tr>
<tr>
<td>$\xi$</td>
<td>Adjustment cost parameter</td>
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</tr>
<tr>
<td>$\lambda$</td>
<td>Financial leverage</td>
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Table 3: Model Comparison

<table>
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<tr>
<th></th>
<th>Data</th>
<th>AA1 (γ = 1.5)</th>
<th>EZ1 (γ = 16.5)</th>
<th>EZ2 (no SV)</th>
<th>AA (γ = 31)</th>
<th>IID</th>
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<tr>
<td><strong>Panel A: Macroeconomic moments</strong></td>
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<td>σ_{Δc} (%)</td>
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<tr>
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<td>0.97</td>
<td>0.97</td>
<td>0.97</td>
<td>0.97</td>
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<tr>
<td>ρ(Δc, Δy)</td>
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</tr>
<tr>
<td><strong>Panel B: Financial moments</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E[R_f] − 1 (%)</td>
<td>1.44</td>
<td>1.88</td>
<td>2.26</td>
<td>1.14</td>
<td>1.60</td>
<td>1.22</td>
</tr>
<tr>
<td>σ(R_f) (%)</td>
<td>1.49</td>
<td>1.38</td>
<td>0.21</td>
<td>0.76</td>
<td>0.93</td>
<td>0.25</td>
</tr>
<tr>
<td>E(R_e − R_f) (%)</td>
<td>6.35</td>
<td>7.10</td>
<td>0.25</td>
<td>6.92</td>
<td>6.56</td>
<td>7.09</td>
</tr>
<tr>
<td>σ(R_e − R_f) (%)</td>
<td>16.85</td>
<td>19.74</td>
<td>13.46</td>
<td>17.56</td>
<td>18.09</td>
<td>15.16</td>
</tr>
<tr>
<td>E(VRP)</td>
<td>12.08</td>
<td>5.26</td>
<td>0.00</td>
<td>1.30</td>
<td>4.25</td>
<td>0.43</td>
</tr>
<tr>
<td>E(VRP) (96-07)</td>
<td>15.15</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>σ(VRP)</td>
<td>34.95</td>
<td>5.78</td>
<td>0.00</td>
<td>1.42</td>
<td>4.40</td>
<td>0.61</td>
</tr>
<tr>
<td>σ(VRP) (96-07)</td>
<td>17.34</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Skewness (VRP)</td>
<td>-4.97</td>
<td>4.12</td>
<td>3.67</td>
<td>3.19</td>
<td>4.06</td>
<td>3.27</td>
</tr>
<tr>
<td>Skewness (VRP) (96-07)</td>
<td>2.76</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Kurtosis (VRP)</td>
<td>35.31</td>
<td>23.66</td>
<td>13.34</td>
<td>14.61</td>
<td>22.40</td>
<td>14.90</td>
</tr>
<tr>
<td>Kurtosis (VRP) (96-07)</td>
<td>10.82</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>σ(M)/E(M)</td>
<td>n.a.</td>
<td>0.83</td>
<td>0.04</td>
<td>0.59</td>
<td>0.83</td>
<td>0.80</td>
</tr>
</tbody>
</table>

This table reports unconditional macroeconomic and financial moments for the benchmark model and alternative models. Model AA1 is the benchmark model with the productivity growth process (1) and the generalized recursive smooth ambiguity utility. Parameter choices are shown in Table 2. Model EZ1 assumes ambiguity neutrality by setting γ = η. Model EZ2 differs from model EZ1 by assuming γ = 16.5. Model AA differs from model AA in that the innovation shock to productivity growth has constant volatility. Model IID assumes IID productivity growth and Epstein-Zin recursive utility with γ = 31. The results are based on 10,000 simulations where each simulation contains 400 quarters of simulated data.
Table 4: Return Predictability

<table>
<thead>
<tr>
<th></th>
<th>1Q</th>
<th>1Y</th>
<th>2Y</th>
<th>3Y</th>
<th>4Y</th>
<th>5Y</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Panel A: Model AA1</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ln(D/P) slope</td>
<td>8.53</td>
<td>24.71</td>
<td>38.54</td>
<td>49.55</td>
<td>58.62</td>
<td>66.49</td>
</tr>
<tr>
<td>$R^2$</td>
<td>2.09</td>
<td>5.34</td>
<td>7.58</td>
<td>9.17</td>
<td>10.27</td>
<td>11.15</td>
</tr>
<tr>
<td>VRP slope</td>
<td>0.11</td>
<td>0.27</td>
<td>0.38</td>
<td>0.46</td>
<td>0.52</td>
<td>0.57</td>
</tr>
<tr>
<td>$R^2$</td>
<td>3.50</td>
<td>6.19</td>
<td>6.81</td>
<td>7.31</td>
<td>7.62</td>
<td>7.83</td>
</tr>
<tr>
<td><strong>Panel B: Model EZ1</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>ln(D/P) slope</td>
<td>2.27</td>
<td>8.20</td>
<td>15.14</td>
<td>21.58</td>
<td>27.59</td>
<td>33.84</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.27</td>
<td>1.02</td>
<td>1.92</td>
<td>2.71</td>
<td>3.40</td>
<td>4.09</td>
</tr>
<tr>
<td>VRP slope</td>
<td>3.84</td>
<td>7.06</td>
<td>8.73</td>
<td>13.09</td>
<td>15.90</td>
<td>17.01</td>
</tr>
<tr>
<td>$R^2$</td>
<td>0.31</td>
<td>0.59</td>
<td>0.77</td>
<td>0.92</td>
<td>0.98</td>
<td>1.02</td>
</tr>
</tbody>
</table>

This table reports return predictability results for models AA1 and EZ1. The dependent variable is the excess equity return. The two independent variables are the log dividend yield ln(D/P) and variance risk premium. Univariate predictive regressions are estimated by the OLS method. The horizon of returns ranges from a quarter to five years. The results are obtained by averaging over 10,000 simulated samples.
Table 5: **Conditional Moments**

<table>
<thead>
<tr>
<th></th>
<th>State 1</th>
<th>State 2</th>
<th>State 3</th>
<th>State 4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>($\mu_l, \sigma_l$)</td>
<td>($\mu_l, \sigma_h$)</td>
<td>($\mu_h, \sigma_l$)</td>
<td>($\mu_h, \sigma_h$)</td>
</tr>
<tr>
<td>Panel A: Model AA1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_{\Delta c}$</td>
<td>0.82</td>
<td>1.47</td>
<td>0.73</td>
<td>1.49</td>
</tr>
<tr>
<td>$\sigma_{\Delta i}$</td>
<td>4.37</td>
<td>6.74</td>
<td>2.63</td>
<td>5.50</td>
</tr>
<tr>
<td>$E[R_f] - 1$</td>
<td>-0.5</td>
<td>0.22</td>
<td>1.64</td>
<td>1.73</td>
</tr>
<tr>
<td>$E(R_e - R_f)$</td>
<td>24.48</td>
<td>25.52</td>
<td>4.78</td>
<td>5.34</td>
</tr>
<tr>
<td>$\sigma(R_e - R_f)$</td>
<td>24.66</td>
<td>29.02</td>
<td>7.86</td>
<td>13.33</td>
</tr>
<tr>
<td>$E(VRP)$</td>
<td>15.50</td>
<td>17.47</td>
<td>2.74</td>
<td>3.35</td>
</tr>
<tr>
<td>$\sigma(M)/E(M)$</td>
<td>1.03</td>
<td>1.03</td>
<td>0.82</td>
<td>0.82</td>
</tr>
<tr>
<td>Panel B: Model EZ1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\sigma_{\Delta c}$</td>
<td>0.94</td>
<td>1.66</td>
<td>0.77</td>
<td>1.66</td>
</tr>
<tr>
<td>$\sigma_{\Delta i}$</td>
<td>3.47</td>
<td>6.09</td>
<td>2.37</td>
<td>4.95</td>
</tr>
<tr>
<td>$E[R_f] - 1$</td>
<td>0.65</td>
<td>0.52</td>
<td>2.12</td>
<td>2.07</td>
</tr>
<tr>
<td>$E(R_e - R_f)$</td>
<td>0.31</td>
<td>0.59</td>
<td>0.16</td>
<td>0.4</td>
</tr>
<tr>
<td>$\sigma(R_e - R_f)$</td>
<td>7.82</td>
<td>10.63</td>
<td>4.26</td>
<td>8.31</td>
</tr>
<tr>
<td>$E(VRP)$</td>
<td>0.03</td>
<td>0.03</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>$\sigma(M)/E(M)$</td>
<td>0.04</td>
<td>0.05</td>
<td>0.02</td>
<td>0.04</td>
</tr>
</tbody>
</table>

This table reports conditional macroeconomic and financial moments for model AA1 and EZ1. Conditional moments are computed by averaging over simulations according to the four productivity growth regimes: low mean and low volatility ($\mu_l, \sigma_l$), low mean and high volatility ($\mu_l, \sigma_h$), high mean and low volatility ($\mu_h, \sigma_l$), and high mean and high volatility ($\mu_h, \sigma_h$).
Table 6: Variance Risk Premium Decomposition

<table>
<thead>
<tr>
<th>Panel A: VRP decomposition</th>
<th>(µ_l, σ_l)</th>
<th>(µ_l, σ_h)</th>
<th>(µ_h, σ_l)</th>
<th>(µ_h, σ_h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>VRP 1st component</td>
<td>5.192</td>
<td>2.652</td>
<td>4.683</td>
<td>-0.613</td>
</tr>
<tr>
<td>VRP 2nd component</td>
<td>0.067</td>
<td>0.004</td>
<td>0.024</td>
<td>0.002</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Panel B: Further decomposition</th>
<th>(µ_l, σ_l)</th>
<th>(µ_l, σ_h)</th>
<th>(µ_h, σ_l)</th>
<th>(µ_h, σ_h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(µ_l, σ_l)</td>
<td>(E_{z_{t+1},t} \left[ \sigma_{R,t+1}^2 \right])</td>
<td>89.817</td>
<td>99.404</td>
<td>19.276</td>
</tr>
<tr>
<td>(\tilde{p}(z_t, z_{t+1}))</td>
<td>0.806</td>
<td>0.028</td>
<td>0.153</td>
<td>0.005</td>
</tr>
<tr>
<td>(p(z_t, z_{t+1}))</td>
<td>0.567</td>
<td>0.019</td>
<td>0.400</td>
<td>0.014</td>
</tr>
<tr>
<td>(µ_l, σ_h)</td>
<td>(E_{z_{t+1},t} \left[ \sigma_{R,t+1}^2 \right])</td>
<td>104.250</td>
<td>110.309</td>
<td>21.610</td>
</tr>
<tr>
<td>(\tilde{p}(z_t, z_{t+1}))</td>
<td>0.019</td>
<td>0.821</td>
<td>0.004</td>
<td>0.156</td>
</tr>
<tr>
<td>(p(z_t, z_{t+1}))</td>
<td>0.013</td>
<td>0.573</td>
<td>0.010</td>
<td>0.404</td>
</tr>
<tr>
<td>(µ_h, σ_l)</td>
<td>(E_{z_{t+1},t} \left[ \sigma_{R,t+1}^2 \right])</td>
<td>66.944</td>
<td>79.072</td>
<td>17.358</td>
</tr>
<tr>
<td>(\tilde{p}(z_t, z_{t+1}))</td>
<td>0.090</td>
<td>0.003</td>
<td>0.876</td>
<td>0.031</td>
</tr>
<tr>
<td>(p(z_t, z_{t+1}))</td>
<td>0.026</td>
<td>0.001</td>
<td>0.941</td>
<td>0.032</td>
</tr>
<tr>
<td>(µ_h, σ_h)</td>
<td>(E_{z_{t+1},t} \left[ \sigma_{R,t+1}^2 \right])</td>
<td>79.606</td>
<td>89.053</td>
<td>18.656</td>
</tr>
<tr>
<td>(\tilde{p}(z_t, z_{t+1}))</td>
<td>0.002</td>
<td>0.091</td>
<td>0.020</td>
<td>0.886</td>
</tr>
<tr>
<td>(p(z_t, z_{t+1}))</td>
<td>0.001</td>
<td>0.026</td>
<td>0.022</td>
<td>0.951</td>
</tr>
</tbody>
</table>

This table presents the variance risk premium decomposition results for the benchmark model. The decomposition takes the form

\[
VRP_t = \sum_{z_{t+1}} E_{z_{t+1},t} \left[ \sigma_{R,t+1}^2 \right] \left[ \tilde{p}(z_t, z_{t+1}) - p(z_t, z_{t+1}) \right] + \sum_{z_{t+1}} \text{cov}_{z_{t+1},t} \left( M_{z_{t+1},t+1} \sigma_{R,t+1}^2 \right) \tilde{p}(z_t, z_{t+1})
\]

Panel A presents the first and second components in different \(z_{t+1}\) states. Panel B displays the terms in the first component conditional on the current state \(z_t\).
<table>
<thead>
<tr>
<th></th>
<th>$\sigma_{\Delta c}$</th>
<th>$\sigma_{\Delta i}$</th>
<th>$\mathbb{E}(R - R_f)$</th>
<th>$\sigma(R - R_f)$</th>
<th>$\mathbb{E}(VRP)$</th>
<th>$\sigma(VRP)$</th>
<th>$\sigma(M)/\mathbb{E}(M)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Benchmark</td>
<td>1.30</td>
<td>5.13</td>
<td>7.10</td>
<td>19.74</td>
<td>5.26</td>
<td>5.78</td>
<td>0.83</td>
</tr>
<tr>
<td>$\eta = 40$</td>
<td>1.26</td>
<td>5.37</td>
<td>14.22</td>
<td>23.30</td>
<td>13.78</td>
<td>11.91</td>
<td>1.54</td>
</tr>
<tr>
<td>$\eta = 25$</td>
<td>1.31</td>
<td>5.01</td>
<td>4.40</td>
<td>17.87</td>
<td>2.50</td>
<td>3.10</td>
<td>0.59</td>
</tr>
<tr>
<td>$\eta = 20$</td>
<td>1.35</td>
<td>4.89</td>
<td>2.58</td>
<td>16.23</td>
<td>1.04</td>
<td>1.41</td>
<td>0.40</td>
</tr>
<tr>
<td>$\gamma = 5$</td>
<td>1.29</td>
<td>5.16</td>
<td>8.40</td>
<td>20.01</td>
<td>5.66</td>
<td>6.05</td>
<td>0.84</td>
</tr>
<tr>
<td>$\gamma = 10$</td>
<td>1.25</td>
<td>5.32</td>
<td>10.50</td>
<td>20.60</td>
<td>6.11</td>
<td>6.23</td>
<td>0.89</td>
</tr>
<tr>
<td>$\psi = 1.2$</td>
<td>1.43</td>
<td>4.32</td>
<td>6.71</td>
<td>19.06</td>
<td>4.83</td>
<td>5.63</td>
<td>0.82</td>
</tr>
<tr>
<td>$\psi = 0.7$</td>
<td>1.81</td>
<td>2.66</td>
<td>5.28</td>
<td>16.77</td>
<td>3.05</td>
<td>4.17</td>
<td>0.79</td>
</tr>
<tr>
<td>$\psi = 0.2$</td>
<td>2.15</td>
<td>2.43</td>
<td>1.81</td>
<td>12.03</td>
<td>0.26</td>
<td>0.20</td>
<td>0.67</td>
</tr>
<tr>
<td>$\xi = 5$</td>
<td>1.04</td>
<td>7.19</td>
<td>6.70</td>
<td>18.95</td>
<td>4.62</td>
<td>5.08</td>
<td>0.83</td>
</tr>
<tr>
<td>$\xi = 1.5$</td>
<td>1.65</td>
<td>3.36</td>
<td>7.25</td>
<td>19.97</td>
<td>5.48</td>
<td>6.14</td>
<td>0.84</td>
</tr>
</tbody>
</table>

This table summarizes the sensitivity analysis results for different parameter values. For each model parameterization, other parameter choices are the same as in the benchmark model. The results are based on 10,000 simulations.
Figure 1: Risk neutral variance and variance risk premium

Panel A: Risk−neutral variance

Panel B: Variance risk premium
Figure 2: Expected productivity growth

![Graph showing productivity growth with conditional mean and productivity growth lines, highlighting periods of 2002 and 2008.](image_url)
Figure 3: VAR impulse responses: data
Figure 4: Cross correlations with output, investment, consumption, and the equity return

Notes: This figure plots the cross-correlograms between the risk neutral variance and output, investment, consumption and the price-dividend ratio, i.e., $\text{Corr}(VRP_t, x_{t+k})$ for $k = -4$ to 4. Output, investment and consumption are in logs and HP-filtered. The figure displays correlations for the historical data and the benchmark model.
Figure 5: Cross correlations with volatilities of output growth, investment growth, consumption growth and equity returns

Notes: This figure plots the cross-correlograms between the risk neutral variance and conditional volatilities of output growth, investment growth, consumption growth and equity returns, i.e., $\text{Corr}(\text{VRP}_t, \sigma_{z,t+k})$ for $k = -4$ to 4. The conditional volatilities are estimated by fitting an AR(1)-GARCH(1,1) model to the historical data and simulated data. The figure displays correlations for the data and the benchmark model.
Notes: This figure plots the impulse response functions for the benchmark model to a negative innovation shock to productivity growth. The economy is assumed to stay in state \((\mu_h, \sigma_l)\) for a long time without the impact of innovation shocks. The innovation shock occurs in the third period and only lasts for that period. The consumption and investment series are detrended.
Figure 7: Impulse responses: mean switching

Notes: This figure plots the impulse response functions for the benchmark model when the expected productivity growth shifts from $\mu_h$ to $\mu_l$. The economy is assumed to stay in state $(\mu_h, \sigma_l)$ for a long time without the impact of innovation shocks. The regime shift occurs in the third period and only lasts for that period. The consumption and investment series are detrended.
Figure 8: **Impulse responses: volatility regime switching**

Notes: This figure plots the impulse response functions for the benchmark model when the conditional volatility of productivity growth shifts from $\sigma_l$ to $\sigma_h$. The economy is assumed to stay in state $(\mu_h, \sigma_l)$ for a long time without the impact of innovation shocks. The regime shift occurs in the third period and only lasts for that period. The consumption and investment series are detrended.
References


